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LOOP VARIABLE INEQUALITIES IN GRAVITY AND GAUGE THEORY

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Abstract.

We point out an incompleteness of formulations of gravitational and gauge theories that use traces of holonomies around closed curves as their basic variables. It is shown that in general such loop variables have to satisfy certain inequalities if they are to give a description equivalent to the usual one in terms of local gauge potentials.

1 Introduction

Since the introduction of an $SL(2, \mathbb{C})$-Yang-Mills connection A as the basic configuration variable for pure gravity by Ashtekar [1,2], the idea of using Wilson loops, i.e. non-local, gauge-invariant quantities depending on the holonomy

$$U_A(\alpha) := \mathrm{P} \exp \oint_\alpha A_\mu d\alpha^\mu \qquad (1.1)$$

around closed curves α in the space(-time) manifold Σ, has found wide application. (The symbol P in (1.1) indicates path ordering along α.) The (normalized) Wilson loop, or traced holonomy, of α is the quantity

$$T_A(\alpha) := \frac{1}{N} \mathrm{Tr}\, U_A(\alpha), \qquad (1.2)$$

with N denoting the dimension of the matrix representation. It is a main ingredient in attempts to quantize the theory non-perturbatively [3] (see also [4] for a recent review). One motivation for adopting a "pure loop approach" is the idea that, although classically equivalent, it may eventually lead to an inequivalent quantum representation, whose non-perturbative features do not resemble those of the quantum connection representation. However, if one wants to reformulate gravity (or non-abelian gauge theory) in terms of these gauge-invariant loop variables, such that the original (gauge-*co*variant) potentials A do not any more appear in the description, one has to make sure that the two formulations are indeed equivalent. This means that, at the kinematical level, there should be a one-to-one correspondence between the gauge potentials $A(x)$ modulo local gauge transformations and the loop variable configurations. In other words, one has to impose appropriate conditions on the set of complex-valued, "bare" loop functions,

$$\{T(\alpha) \,|\, \alpha \text{ a closed curve in } \Sigma\}, \qquad (1.3)$$

in order to ensure they are the traces of holonomies of a given gauge group G. Finding a complete set of such conditions involves a number of subtleties, and for general gauge group G is an unsolved problem.

On the one hand, there are the so-called Mandelstam constraints, certain G-dependent, algebraic constraints among the loop variables, which have been discussed by several authors

[5,6,7,8]. The Mandelstam constraints for $SL(2, \mathbb{C})$ (and for any of its subgroups) are given by

$$T(\alpha)T(\beta) - \frac{1}{2}\big(T(\alpha \circ_x \beta) + T(\alpha \circ_x \beta^{-1})\big) = 0. \tag{1.4}$$

The notation in equation (1.4) refers to a configuration of two loops α and β intersecting in a point x, the base point for all loops, with $\circ_x$ denoting the loop composition in x. Polynomial constraints of this kind lead to an overcompleteness of the set of loop functions. There are several possibilities of implementing them either classically or in the quantum theory. A maximal identification of the independent physical degrees of freedom seems to be possible only in the regularized lattice version of the theory [9]. An algebraic treatment of the Mandelstam constraints for $SL(2, \mathbb{C})$ has been given by Ashtekar and Isham [10], who construct an algebra of T-variables in such a way that the constraints (1.4) are satisfied automatically and continue to hold in any quantum representation of that algebra.

On the other hand, there is the problem of selecting an appropriate set of closed curves on which the loop variables are to be defined. It turns out that the fundamental objects are not the loops themselves, but certain equivalence classes of loops (under reparametrization, retracing, orientation reversal etc., see [11] for a detailed discussion and references). Since the distinction is not essential in the present context, I will continue to use the word 'loop' for such an equivalence class.

Moreover, there are examples of gauge groups where the traced holonomies do not contain all the local gauge-invariant information. For example, there are non-trivial subgroups of "null rotations" of $SL(2, \mathbb{C})$ which are mapped into a single configuration in terms of the variables $T(\alpha)$ [12].

The purpose of this paper is to point out that, in addition to the well-known Mandelstam constraints, certain inequalities have to be satisfied by the loop variables $T(\alpha)$ in order to achieve equivalence with the usual connection representation. These inequalities cannot be derived from the Mandelstam constraints. This renders quantization approaches based on loop variable algebras, and which are constructed independently of quantum representations based on the connection A, incomplete.

Because of its relevance for gravitational theories. I will discuss the case of gauge group $G = SL(2, \mathbb{C})$ and two of its subgroups ($SU(2)$ and $SU(1,1)$), in the defining representation by 2×2 complex matrices. However, similar problems are expected to occur for other non-abelian gauge groups and representations. - The next section contains the derivation of some

specific examples of inequalities between the loop variables. The consequences of the present result for general (quantum) loop approaches are outlined in Sec.3.

2 Loop inequalities for subgroups of $SL(2, \mathbb{C})$

The "gauge group" for the gravitational field in Ashtekar's Hamiltonian reformulation is the non-compact Lie group $SL(2, \mathbb{C})$. However, the physical configurations take their values in a particular $SO(3)$-subgroup of $SL(2, \mathbb{C})$, which is projected out by a set of reality conditions on phase space. Unfortunately, no manageable form for these reality conditions is known in the loop formulation, where the basic variables are given by the Wilson loops (1.2).

In the following I will look at two other subgroups of $SL(2, \mathbb{C})$, one of type $SU(2)$, and the other of type $SU(1, 1)$, which are embedded into $SL(2, \mathbb{C})$ in a simple way. The former is of course of interest for Yang-Mills theory, whereas the latter appears in reduced gravitational models such as 2+1 gravity (see [13] for a treatment in the loop formulation), and 3+1 gravity with one (space-like) Killing vector field.

The new inequalities that have to be satisfied by the loop variables describing these subgroups become apparent only when one uses the original explicit representation of the 2×2 holonomy matrices. From these one derives inequalities between the group parameters, which then translate into gauge-invariant inequalities among the loop variables.

We will first treat the case of $SU(2) \subset SL(2, \mathbb{C})$, given by the subgroup of matrices of the form

$$U_\alpha = \begin{pmatrix} \alpha_1 + i\alpha_2 & \alpha_3 + i\alpha_4 \\ -\alpha_3 + i\alpha_4 & \alpha_1 - i\alpha_2 \end{pmatrix}, \; U_\beta = \begin{pmatrix} \beta_1 + i\beta_2 & \beta_3 + i\beta_4 \\ -\beta_3 + i\beta_4 & \beta_1 - i\beta_2 \end{pmatrix}, \; \text{etc.,} \qquad (2.1)$$

with real parameters α_i, β_i, ... , subject to the constraints $\sum_i \alpha_i^2 = 1$, $\sum_i \beta_i^2 = 1$, etc.. The indices α and β on the holonomy matrices U label loops starting and ending at a given base point x. In this parametrization, the group variables α_i can be thought of as the components of a unit vector $\vec{\alpha}$ imbedded into $\mathbb{R}^4$. The relevant traced holonomies are

$$T(\alpha) = \alpha_1$$
$$T(\alpha \circ_x \beta) = \alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3 - \alpha_4\beta_4 =: \alpha_1\beta_1 - \vec{\alpha}_\perp \cdot \vec{\beta}_\perp, \qquad (2.2)$$

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where a convenient vectorial notation has been introduced for the 2-, 3- and 4-components of the vectors $\vec{\alpha}$ and $\vec{\beta}$. The conditions $\sum_i \alpha_i^2 = 1$ imply the well-known fact that the loop variables $T(\alpha)$ for $G = SU(2)$ are bounded functions on loop space, namely,

$$-1 \leq T(\alpha) \leq 1, \qquad \forall \alpha. \tag{2.3}$$

This boundedness was used in a crucial way by Ashtekar and Isham in their application of Gel'fand spectral theory to the abelian algebra of T-variables. However, there exist more complicated inequalities, which so far seem to have been overlooked. For the case of $SU(2)$, they are easily derived from geometric considerations, but in view of less straightforward cases (such as the one of $SU(1,1)$ presented below), I will sketch their proper mathematical derivation.

I will show that, given arbitrary values for the traced holonomies around two loops α and β, $T(\alpha) = c$, $T(\beta) = c'$, say, with $c, c' \in [-1, 1]$, the Wilson loop $T(\alpha \circ_x \beta)$ cannot assume arbitrary values in the interval $[-1, 1]$, but rather has to obey an inequality, depending on c and c'. For this purpose, it is convenient (but not necessary) to consider linear combinations of the form

$$L_2(\alpha, \beta) := \frac{1}{2} \big(T(\alpha \circ_x \beta^{-1}) - T(\alpha \circ_x \beta) \big) = \vec{\alpha}_\perp \cdot \vec{\beta}_\perp, \tag{2.4}$$

which were first introduced in [14]. (Note that the *sum* of the two terms appearing on the right-hand side of the definition, $T(\alpha \circ_x \beta^{-1}) + T(\alpha \circ_x \beta)$, by virtue of (1.4) is an algebraically dependent quantity.) In order to find the (conditional) extrema of $L_2(\alpha, \beta)$, for fixed $T(\alpha)$ and $T(\beta)$, one best uses the method of Lagrangian multipliers. Extremal points in $\mathbb{R}^8$ (with coordinates α_i, β_i, $i = 1, \ldots, 4$) must satisfy

$$\nabla L_2(\alpha, \beta) - \lambda \nabla(\alpha_1 - c) - \lambda' \nabla(\beta_1 - c') - \mu \nabla(\sum_i \alpha_i^2 - 1) - \mu' \nabla(\sum_i \beta_i^2 - 1) = 0, \quad (2.5)$$

where λ, λ', μ and μ' are real Lagrange multipliers. From (2.5) one derives the condition $\vec{\alpha}_\perp = k \vec{\beta}_\perp$, $k \in \mathbb{R} \backslash \{0\}$, for local extrema. To determine the nature of these extrema, one has to compute the Hessian in these points (in a suitable local basis of tangent vectors), and its signature. This computation yields an absolute maximum in points with $\vec{\alpha}_\perp = k \vec{\beta}_\perp$,

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$k = \frac{\sqrt{1-\alpha_1^2}}{\sqrt{1-\beta_1^2}}$, and an absolute minimum in points with $\vec{\alpha}_\perp = k\vec{\beta}_\perp$, $k = -\frac{\sqrt{1-\alpha_1^2}}{\sqrt{1-\beta_1^2}}$. Rephrased in terms of loop variables, the result reads as follows: *for the SU(2)-subgroup given by matrices of the form (2.1), and for given $T(\alpha)$ and $T(\beta)$, $L_2(\alpha, \beta)$ may only assume values in the interval*

$$-\sqrt{(1-T(\alpha)^2)(1-T(\beta)^2)} \le L_2(\alpha, \beta) \le \sqrt{(1-T(\alpha)^2)(1-T(\beta)^2)}. \tag{2.6}$$

The equivalent statement in terms of the original traced holonomies is obtained by substituting L_2 according to (2.4).

Let us now turn to the case of $SU(1,1) \subset SL(2, \mathbb{C})$, given by the subgroup of matrices of the form

$$U_\alpha = \begin{pmatrix} \alpha_1 + i\alpha_2 & \alpha_3 + i\alpha_4 \\ \alpha_3 - i\alpha_4 & \alpha_1 - i\alpha_2 \end{pmatrix}, \; U_\beta = \begin{pmatrix} \beta_1 + i\beta_2 & \beta_3 + i\beta_4 \\ \beta_3 - i\beta_4 & \beta_1 - i\beta_2 \end{pmatrix}, \; \text{etc.,} \tag{2.7}$$

again with real parameters α_i, β_i, ... , but now subject to the constraints $\alpha_1^2 + \alpha_2^2 - \alpha_3^2 - \alpha_4^2 = 1$, $\beta_1^2 + \beta_2^2 - \beta_3^2 - \beta_4^2 = 1$, etc.. As a consequence of the non-compactness of $SU(1,1)$, the loop variables $T(\alpha)$ are no longer bounded functions on loop space. Adopting the same definition for the variables L_2 as in (2.4) above leads to

$$L_2(\alpha, \beta) = \alpha_2\beta_2 - \alpha_3\beta_3 - \alpha_4\beta_4. \tag{2.8}$$

Note that, as in the case of $SU(2)$, all the traced holonomies $T(\alpha)$ are *real* functions on loop space. Following the same strategy as for $SU(2)$ leads to the condition

$$\nabla L_2(\alpha, \beta) - \lambda\nabla(\alpha_1 - c) - \lambda'\nabla(\beta_1 - c') - \mu\nabla(\alpha_1^2 + \alpha_2^2 - a_3^2 - a_4^2 - 1) - \mu'\nabla(\beta_1^2 + \beta_2^2 - \beta_3^2 - \beta_4^2 - 1) = 0 \tag{2.9}$$

for local extrema. Again this yields as solutions points with $\vec{\alpha}_\perp = k\vec{\beta}_\perp$, $k \in \mathbb{R}\backslash\{0\}$. The computation of the Hessian is slightly more involved, and one finds that its signature is not necessarily positive or negative (semi-)definite. The final result is as follows: $L_2(\alpha, \beta)$ has a local maximum in points for which $\vec{\alpha}_\perp = k\vec{\beta}_\perp$, $k = -\frac{\sqrt{1-\alpha_1^2}}{\sqrt{1-\beta_1^2}} < 0$, and $1 - T(\alpha)^2 \ge 0$

($\iff$ $1 - T(\beta)^2 \geq 0$), and a local minimum in points with $\vec{\alpha}_\perp = k\vec{\beta}_\perp$, $k = \frac{\sqrt{1-\alpha_1^2}}{\sqrt{1-\beta_1^2}} > 0$, and $1 - T(\alpha)^2 \geq 0$ ($\iff$ $1 - T(\beta)^2 \geq 0$). All other extrema are saddle points. Again this result may be restated in a gauge-invariant way in terms of the loop variables only: *for the $SU(1,1)$-subgroup given by matrices of the form (2.7), and for given $T(\alpha)$ and $T(\beta)$, and if both $1 - T(\alpha)^2 \geq 0$ and $1 - T(\beta)^2 \geq 0$, $L_2(\alpha, \beta)$ may only assume values*

$$L_2(\alpha, \beta) \leq -\sqrt{(1 - T(\alpha)^2)(1 - T(\beta)^2)} \quad \text{or} \quad L_2(\alpha, \beta) \geq \sqrt{(1 - T(\alpha)^2)(1 - T(\beta)^2)}. \quad (2.10)$$

For other values of $T(\alpha)$ and $T(\beta)$ there are no restrictions, and $L_2(\alpha, \beta)$ may assume any value on the real line.

3 Conclusions

In the preceding section we derived some examples of inequalities that have to be satisfied by loop functions $T(\alpha)$ in order that they correspond to the traced holonomies of particular subgroups of $SL(2, \mathbb{C})$. These inequalities are yet another manifestation of the non-linearities of the underlying physical configuration spaces. Incidentally, our derivation provides an answer to a question posed in [10], namely, how to distinguish between the subgroups $SU(2)$ and $SU(1,1)$ of $SL(2, \mathbb{C})$, without making reference to the gauge potentials A.

The question remains of how such inequalities, whenever they appear, can be incorporated in the loop formulation of gravitational and gauge theories. It can easily be shown that inequalities are present even if one considers only loop configurations without (self-)intersections. This means that such configurations do not just represent isolated points in the configuration space, but are abundant. Furthermore, inequalities such as (2.6) and (2.10) do not exhaust the set of all restrictions for a given gauge group, as is exemplified by the loop treatment of the 1×1-lattice gauge theory [8]. They depend in an essential way on the geometry of the loop configuration, and there is no obvious systematic way of identifying all of them, unless one works in a lattice regularization where the number of loop variables can be greatly reduced.

This observation is relevant for 3+1-dimensional gravity insofar as also there the physical data take values in a subalgebra of $sl(2, \mathbb{C})$, namely, in $so(3)$. The embedding of the subalgebra is much more complicated in this case, since it is defined in terms of non-linear equations not on the configuration, but on the *phase* space of the canonical variable pairs

(A, E). The examples treated above suggest that even if we were to find a way to implement this restriction to $so(3)$ in the loop representation (for example, by finding a suitable form of the reality conditions in terms of loop variables), we may still have to deal with additional inequalities of the kind described in Sec.2 on the thus reduced space of loop variables.

Inequalities are notoriously hard to implement in canonical quantization procedures, and their importance for gravity in the metric representation (where one requires $\det g_{ij} > 0$) has repeatedly been emphasized by Isham (see, for example, [15]). A similar problem may appear in the loop formulation of 3+1 gravity, unless one wants to rely on a loop transform of quantum states (along the lines discussed in [10]) from the connection representation to the loop representation, whose existence however has not been established. The presence of loop variable inequalities is directly relevant for pure loop formulations of Yang-Mills theory and various reduced gravitational models. This observation may not come entirely as a surprise, since in the case of Yang-Mills theory it is well-known that inequalities appear in certain non-standard parametrizations of the theory (see, for example, the flux representation discussed in [16]). - Unless the existence of such loop variable inequalities is taken properly into account, any formulation of gravitational and gauge theories based exclusively on the Wilson loops (1.2) remains inequivalent to its usual formulation in terms of connection variables.

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